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Is Vegas-Style Solitaire Fair?

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Abstract: We explore the version of Vegas-style Solitaire most commonly found on personal computers. There is a $52 fee to play the game with each card in the "suit stack" at the end of the game paying $5. We analyze empirical data to explore the fairness of the game.

1 Introduction

The version of Solitaire under consideration involves a single deck of 52 cards. Play begins with a "completely shuffled" deck. For details regarding the number of shuffles needed to achieve this goal, and further complicating factors, refer to [4]. The results in this landmark paper deal with the use of real cards shuffled by real people and indicate that seven riffles are needed to remove detectable non-randomness from a new deck of cards. Estimates on the number of riffles needed to achieve the same goal in computer simulations range as high as nine, but in order to gather a large amount of data in a short time most of our games were actually played on computers. We chose the version in which three cards are dealt at a time; the game ends either after all cards end up in the suit stacks or we go through the deck three times, whichever is first.

Vegas-style Solitaire involves, as must be expected, a fee to play and a payoff scheme. In our preliminary analysis, the fee to play will be $52; each card that is in the "suit stacks" at the end of the game pays $5. We will consider a game a *win* if the net payoff is positive; otherwise the game is a *loss*. We call a game a *complete win* if all 52 cards make it to the suit stacks; the net payoff in this case is $208.

2 What Does "Fair" Mean?

A *random experiment* is a procedure that ends in an outcome that cannot be determined with complete certainty before the experiment is performed. However, every possible outcome can be described or even explicitly listed. The collection of all outcomes is called the *sample space* of the experiment.

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A *random variable* is a function that assigns a numerical value to each element of a sample space. A *discrete random variable* is a variable that may take any of a finite or countable set of values, each with a corresponding probability. Random variables are typically denoted with a letter such as X; the values of the variable are denoted x_i and the corresponding probabilities are p_i. Probabilities based on observation of repeated trials of an experiment are known as *empirical* probabilities.

The *expected value* of a discrete random variable X that takes values x_i with probability p_i is given by $E(X) = \sum_i p_i x_i$. A game involving the exchange of money is *fair* if the expected value is \$0.00. See Chapter 3 of [3] for a more detailed description of mathematical expectation.

Example 1 *Consider a game that has a cost of* \$1 *to play. A number between 1 and 6 is chosen, then a die is rolled. If the chosen number is rolled then the player receives* \$6*, for a net gain of* \$5*.*

Notice that even though many more games result in losses than wins, this game is fair since the expected *monetary payoff* over time is \$0.00.

3 Complicating Factors

The skill level of a player certainly can have an effect on the outcome of a game of Solitaire. An experienced or observant player may notice moves that another player overlooks. It has been interesting to observe the results when multiple players use different decks of cards that have been arranged in precisely the same order. Some players lose while others break even or win.

The energy level or sobriety (or lack thereof!) of a player can also have an effect on the results of a game. After several consecutive games a player may not notice certain 'good' moves.

Solitaire involves a 'greedy' algorithm concept similar to that found in solving certain graph theory optimization problems. When playing the game, a player must at times make choices using only limited knowledge. As a result, the player must make the 'best' available choice. The collection of these choices may or may not result in a winning game.

4 Analysis of the Data

We turn now to an empirical analysis of Solitaire. While playing the game, we kept a record of the number of cards in the suit stacks at the end of each game as well as a running total of the money won or lost.

4.1 What Can Happen?

No reasonable player will stop a game with 51 cards in the suit stacks. If only one card is left, it must be a king and is playable. Since there are no rules to the contrary, a card can be brought back into play after being placed in the suit stacks. With this in mind, consider a worst-case scenario that can occur if there are 48 cards in the suit stack. Without loss of generality, suppose the four cards left are $K\heartsuit$, $Q\heartsuit$, $J\heartsuit$ and $10\heartsuit$ and that the 10 is buried under the jack, which is under the queen with only the king visible. The goal is to make the 10 visible so that it can be played on $9\heartsuit$ which is already in the suit stack. Once this happens, all the other cards can then be moved to the suit stacks in order. To achieve this, move the king to an available "open position" which makes the queen visible. It is now possible to bring $K\spadesuit$ back into play so $Q\heartsuit$ can be played on it. This makes $J\heartsuit$ visible. Now $K\clubsuit$ can be brought back into play, which allows us to play $Q\clubsuit$ on $K\heartsuit$. Now play $J\heartsuit$ on $Q\clubsuit$ to make $10\heartsuit$ visible.

Claim 1 *Short of a complete win, no reasonable player will stop the game with more than 47 cards in the suit stacks.*

4.2 Probability of Outcomes

We played 1250 games of Solitaire in order to establish the empirical discrete distribution given below. The integers represent the number of cards in the suit stacks at the end of a game. As a result of Claim 1 there are 49 possible outcomes and empirical probabilities; only 33 of these actually showed up in the games played. With more data, some of the other possible values would likely appear.

0	1	2	3	4	5	6	7
47/1250	94/1250	128/1250	153/1250	116/1250	130/1250	112/1250	103/1250
8	9	10	11	12	13	14	15
68/1250	64/1250	36/1250	33/1250	25/1250	25/1250	8/1250	8/1250
16	17	18	19	20	21	22	23
11/1250	3/1250	5/1250	5/1250	2/1250	5/1250	1/1250	1/1250
24	25	26	27	28	29	32	45
3/1250	1/1250	1/1250	1/1250	2/1250	1/1250	1/1250	1/1250

The empirical probability of a complete win with all 52 cards in the suit stacks was 56/1250. Based on these 1250 games, the expected value of the number of cards in the suit stacks at the end of a game is

$$E(X) = \sum_i p_i \cdot x_i = 9842/1250 = 7.8736. \tag{1}$$

5 How Can the Game Be Made Fair?

Observation 1 *With the described payoff scheme, an average of 10.4 cards must reach the suit stacks in order for the game to be fair.*

By Equation 1, we can expect about 7.9 cards to reach the suit stacks. In other words, the expected loss per game under the described payoff scheme is about $7.9 \cdot 5 - 52 = -\$12.50$. An interesting problem for future consideration is to compare this expected loss per game to the expected value of other popular games such as craps, roulette or blackjack.

One way to make the game fair is to change the cost to play the game. Since we can expect about $7.9 \cdot 5 = \$39.50$ on each play, a fair cost would be \$39.50.

Since the \$52 cost to play corresponds so well with the 52 cards in a deck, we could also change the payoff scheme. If x represents the payoff value for each card in the suit stacks at the end of the game, we need to solve the equation $7.9x - 52 = 0$. This results in a payoff of about \$6.58 for each card.

Other rules of the game could be changed as well. For instance, rather than dealing three cards at a time and going through the deck three times, we could deal one card at a time and go through the deck as many times as we like.

6 Concluding Remarks

There are many questions and directions of further research left open in this investigation. We have proceeded empirically and have (for good reason, we think) completely avoided the pursuit of a closed-form analysis of Solitaire. It may be interesting to write a computer program that simulates the game. Another question is whether there exist certain deals that will not permit a single card to make it to the suit stacks even with perfect play. We invite all interested parties to join us for further exploration.

7 Acknowledgement

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References

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